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Periodic motions of the Kowalevski gyrostat in two constant fields

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Abstract
The case of the gyrostat motion found by A G Reyman and M A Semenov-Tian-Shansky is known as the Liouville integrable Hamiltonian system with three degrees of freedom without symmetry groups. We find the set of points at which the integral map has rank 1. This set consists of special periodic motions generating the singular points of bifurcation diagrams on iso-energetic surfaces. For such motions, all phase variables are expressed as algebraic functions of one auxiliary variable satisfying the differential equation integrable in elliptic functions of time. It is shown that the corresponding points in three-dimensional space of the integral constants belong to the intersection of two sheets of the discriminant surface of the Lax curve.

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1. Introduction

The equations of motion of a gyrostat with a fixed point in two constant fields (say, gravitational and magnetic) referred to the moving frame have the form

$$\begin{aligned}&\boldsymbol{I}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = (\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{\lambda}) \times \boldsymbol{\omega} + \boldsymbol{r}_1 \times \boldsymbol{\alpha} + \boldsymbol{r}_2 \times \boldsymbol{\beta},\\ &\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \qquad \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}t} = \boldsymbol{\beta} \times \boldsymbol{\omega}.\end{aligned} \tag{1}$$

Here $\boldsymbol{\omega}$ is the angular velocity, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are fixed in space fields intensity vectors. Constant (with respect to the body) vectors $\boldsymbol{r}_1, \boldsymbol{r}_2$ denote radius vectors of the centres of fields application, $\boldsymbol{\lambda}$ stands for the gyrostatic momentum, $\boldsymbol{I}$ is the inertia tensor at the fixed point O. The restriction of such a system to any non-degenerate common level of three geometrical integrals (Casimir functions $|\boldsymbol{\alpha}|^2, |\boldsymbol{\beta}|^2, \boldsymbol{\alpha} \cdot \boldsymbol{\beta}$) in $\mathbb{R}^9(\boldsymbol{\omega}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is Hamiltonian with three degrees of freedom.

Let $Oe_1e_2e_3$ be the orthonormal basis of the principal axes of inertia. Suppose that the principal inertia moments satisfy the Kowalevski ratio 2:2:1, the gyrostatic momentum is directed along the dynamic symmetry axis $\boldsymbol{\lambda} = \lambda e_3$ ($\lambda = \text{const}$), and the vectors $\boldsymbol{r}_1, \boldsymbol{r}_2$ are parallel to the equatorial plane Oe_1e_2. In the work [1] the complete Liouville integrability of such system (1) has been proved. Despite this fact, for the case $\lambda \neq 0$ the only explicit integrations or qualitative investigations of (1) known up-to-date [2–4] deal with the axially symmetric force ($\boldsymbol{\beta} \equiv 0$). We suppose that

$$\boldsymbol{\alpha} \times \boldsymbol{\beta} \neq 0, \qquad \boldsymbol{r}_1 \times \boldsymbol{r}_2 \neq 0.$$

The first condition implies that the geometrical integrals are independent and the corresponding common level P^6 is diffeomorphic to $SO(3) \times \mathbb{R}^3$.

For each common level P^6 there exists a linear change of variables, which after a proper choice of the moving frame gives $\boldsymbol{r}_1 = \boldsymbol{e}_1, \boldsymbol{r}_2 = \boldsymbol{e}_2, \boldsymbol{\alpha} \perp \boldsymbol{\beta}$ [5]. If in addition it happens that $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, we obtain the case of Yehia [6]; the system admits an S^1-symmetry and therefore is reducible to a family of systems with two degrees of freedom. In the following we deal with the irreducible case $|\boldsymbol{\alpha}| \neq |\boldsymbol{\beta}| \neq 0$. Thus, without loss of generality we consider the geometrical integrals

$$|\boldsymbol{\alpha}|^2 = a^2, \qquad |\boldsymbol{\beta}|^2 = b^2, \qquad \boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0 \tag{2}$$

with arbitrary constants

$$a > b > 0. \tag{3}$$

Choose the measurement units to obtain $\boldsymbol{I} = \text{diag}\{2, 2, 1\}$. Then the three involutive first integrals on the level (2) are

$$\begin{aligned} H &= \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - \alpha_1 - \beta_2 - \frac{\lambda^2}{2}, \\ K &= \left(\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2\right)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2 \\ &\qquad + 2\lambda\left[(\omega_3 - \lambda)\left(\omega_1^2 + \omega_2^2\right) + 2\omega_1\alpha_3 + 2\omega_2\beta_3\right], \\ G &= \frac{1}{4}\left(M_\alpha^2 + M_\beta^2\right) + \frac{1}{2}(\omega_3 - \lambda)M_\gamma - b^2\alpha_1 - a^2\beta_2. \end{aligned} \tag{4}$$

Here $M_\alpha = (\boldsymbol{I\omega} + \boldsymbol{\lambda}) \cdot \boldsymbol{\alpha}$, $M_\beta = (\boldsymbol{I\omega} + \boldsymbol{\lambda}) \cdot \boldsymbol{\beta}$, $M_\gamma = (\boldsymbol{I\omega} + \boldsymbol{\lambda}) \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta})$.

The constant term $(-\lambda^2/2)$ in the energy integral is introduced for compatibility with the work [1] and more detailed paper [7]. Note that the integral K for the Kowalevski gyrostat in two constant fields was first found by Yehia [6], and the integral G generalizing the square of the classical cyclic integral was found by Reyman and Semenov-Tian-Shansky in the work [1] by means of the Lax representation for the system (1).

Introduce the integral map

$$J = H \times K \times G : P^6 \to \mathbb{R}^3. \tag{5}$$

The inverse images $J^{-1}(h, k, g)$ for all $(h, k, g) \in \mathbb{R}^3$ define the Liouville foliation of P^6. It is likely that the system (1) is non-degenerate on P^6 as the Hamiltonian system with three degrees of freedom. Then periodic trajectories on resonance tori, though filling a set of measure zero, are dense in P^6. Nevertheless, such trajectories are of no interest from the point of view of analytical integration or topological analysis of the system. The topological analysis, in the case of two degrees of freedom, is based on the study of bifurcation diagrams of the integral maps and corresponding transformations of two-dimensional Liouville tori [8]. In the aspect of the Liouville equivalence, more analysis of the singular points (the so-called nodes) of these diagrams is needed [9]. In our case consider the bifurcation diagram $\Sigma(J) \subset \mathbb{R}^3$ as a two-dimensional cell complex. Then the singular points form the union of skeletons of dimensions

0 and 1. The 0-skeleton is the image of the equilibria in the system; it is shown in [10] that there exist exactly four equilibria on each P^6 satisfying (3) (see also [11] for related results). The question of finding 1-cells of $\Sigma(J)$ is much more complicated. The corresponding values of the first integrals are generated by periodic trajectories that are Liouville tori themselves, i.e., closed orbits satisfying rank $J = 1$. Let us call such a trajectory a special periodic motion (SPM). Another way to define an SPM is to say that it is a one-dimensional orbit of the Poisson action on P^6 generated by the involutive set of the first integrals [9].

For the classical case of the Kowalevski top in the gravity field all SPMs are permanent rotations around the vertical axis. For the Kowalevski top in two constant fields ($\lambda = 0$) the set of SPMs, as shown in [12], consists of three families of pendulum motions pointed out in [5] for an arbitrary rigid body and the families of critical periodic motions of the Bogoyavlensky case [13]. These last motions were first described in [14] and explicitly integrated in [15]. It is easy to check that in the case $\lambda \neq 0$ only the following pendulum motions remain:

$$\begin{aligned} &\boldsymbol{\alpha} = a(\boldsymbol{e}_1\cos\varphi - \boldsymbol{e}_2\sin\varphi), \qquad \boldsymbol{\beta} = \pm b(\boldsymbol{e}_1\sin\varphi + \boldsymbol{e}_2\cos\varphi), \\ &\boldsymbol{\alpha}\times\boldsymbol{\beta} \equiv \pm ab\boldsymbol{e}_3, \qquad \boldsymbol{\omega} = \dot\varphi\boldsymbol{e}_3, \qquad \ddot\varphi = -(a\pm b)\sin\varphi. \end{aligned} \tag{6}$$

Note that these motions were first found by Yehia [16] with no conditions imposed on the moments of inertia but under some special restrictions for the centers of fields application.

The values of the first integrals (4) at solutions (6) satisfy the following:

$$k = (a\pm b)^2, \qquad g = \mp abh. \tag{7}$$

The inequalities for h can easily be derived for any combination of signs. The admissible values among (7) are obviously included in the 1-skeleton of the bifurcation diagram. In this paper, in addition to solutions (6), we find all special periodic motions of the Kowalevski gyrostat in two constant fields. We obtain the expressions for the corresponding points in the image of the integral map filling the remaining part of the 1-skeleton of the bifurcation diagram. We also prove that these points on the discriminant surface of the Lax algebraic curve provide the pair-wise intersection of two-dimensional sheets.

2. Bifurcation surfaces

Introduce the change of variables ($\mathrm{i}^2 = -1$):

$$\begin{aligned} &x_1 = (\alpha_1-\beta_2) + \mathrm{i}(\alpha_2+\beta_1), && x_2 = (\alpha_1-\beta_2) - \mathrm{i}(\alpha_2+\beta_1), \\ &y_1 = (\alpha_1+\beta_2) + \mathrm{i}(\alpha_2-\beta_1), && y_2 = (\alpha_1+\beta_2) - \mathrm{i}(\alpha_2-\beta_1), \\ &z_1 = \alpha_3 + \mathrm{i}\beta_3, && z_2 = \alpha_3 - \mathrm{i}\beta_3, \\ &w_1 = \omega_1 + \mathrm{i}\omega_2, && w_2 = \omega_1 - \mathrm{i}\omega_2, \\ &w_3 = \omega_3. \end{aligned} \tag{8}$$

Then from (1),

$$\begin{aligned} &x_1' = z_1w_1 - x_1w_3, && x_2' = x_2w_3 - z_2w_2, \\ &y_1' = z_2w_1 - y_1w_3, && y_2' = y_2w_3 - z_1w_2, \\ &z_1' = \tfrac12(x_1w_2 - y_2w_1), && z_2' = \tfrac12(y_1w_2 - x_2w_1), \\ &w_1' = -\tfrac12[w_1(w_3-\lambda) + z_1], && w_2' = \tfrac12[w_2(w_3-\lambda)+z_2], \\ &w_3' = \tfrac12(y_2-y_1). \end{aligned} \tag{9}$$

Here the stroke stands for $\mathrm{d}/\mathrm{d}(\mathrm{i}t)$. The geometrical integrals (2) take the form

$$z_1^2 + x_1 y_2 = r^2, \qquad z_2^2 + x_2 y_1 = r^2, \tag{10}$$

$$x_1 x_2 + y_1 y_2 + 2 z_1 z_2 = 2p^2, \tag{11}$$

where the positive constants $p = \sqrt{a^2 + b^2}$ and $r = \sqrt{a^2 - b^2}$ are introduced according to the conditions (3). Rewrite (4) as follows:

$$\begin{aligned}
H &= w_1 w_2 + \tfrac{1}{2} w_3^2 - \tfrac{1}{2}(y_1 + y_2 + \lambda^2),\\
K &= \left(w_1^2 + x_1\right)\left(w_2^2 + x_2\right) + 2\lambda(w_1 w_2 w_3 + z_2 w_1 + z_1 w_2) - 2\lambda^2 w_1 w_2,\\
G &= \tfrac{1}{4}(p^2 - x_1 x_2) w_3^2 + \tfrac{1}{2}(x_2 z_1 w_1 + x_1 z_2 w_2) w_3\\
&\quad + \tfrac{1}{4}(x_2 w_1 + y_1 w_2)(y_2 w_1 + x_1 w_2) - \tfrac{1}{4} p^2 (y_1 + y_2) + \tfrac{1}{4} r^2 (x_1 + x_2)\\
&\quad + \tfrac{1}{2}\lambda(z_1 z_2 w_3 + y_2 z_2 w_1 + y_1 z_1 w_2) + \tfrac{1}{4}\lambda^2(p^2 - y_1 y_2).
\end{aligned} \tag{12}$$

The Lax representation found in the work [1] in our notation has the form

$$L' = LM - ML, \tag{13}$$

where

$$L = \frac{1}{\varkappa}\begin{pmatrix} 0 & x_2 & 0 & z_2\\ -x_1 & 0 & -z_1 & 0\\ 0 & z_2 & 0 & -y_1\\ -z_1 & 0 & y_2 & 0 \end{pmatrix} + 2\begin{pmatrix} \lambda & 0 & -w_2 & 0\\ 0 & -\lambda & 0 & w_1\\ -w_1 & 0 & -w_3 & -2\varkappa\\ 0 & w_2 & 2\varkappa & w_3 \end{pmatrix},$$

$$M = \frac{1}{2}\begin{pmatrix} -w_3 & 0 & w_2 & 0\\ 0 & w_3 & 0 & -w_1\\ w_1 & 0 & w_2 & 2\varkappa\\ 0 & -w_2 & -2\varkappa & -w_3 \end{pmatrix},$$

$\varkappa$ is the spectral parameter, the derivative in (13) is calculated in virtue of (9). The eigenvalue equation $\det(L - \zeta E) = 0$ defines the algebraic curve associated with this representation [7]. Put $s = 2\varkappa^2$ and denote by h, k, g the arbitrary constants of the integrals (12). The equation of the algebraic curve becomes

$$\begin{aligned}
\Gamma(s,\zeta) = \zeta^4 - \frac{4}{s}[p^2 - 2(h + \lambda^2)s + 2s^2]\zeta^2 + \frac{4}{s^2}[r^4 + 4(2g - p^2 h - p^2\lambda^2)s\\
+ 4(k + 2\lambda^2 h + \lambda^4)s^2 - 8\lambda^2 s^3] = 0.
\end{aligned} \tag{14}$$

Suppose that (s, ζ) is a singular point of this curve considered as a subset in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$. It is easily checked that in the case $\lambda \neq 0$ the points with $s = \infty$ or $\zeta = \infty$ are always regular for the map Γ. Then the singularity conditions have the form

$$\frac{\partial \Gamma}{\partial s} = 0, \qquad \frac{\partial \Gamma}{\partial \zeta} = 0. \tag{15}$$

Elimination of ζ in (14), (15) leads to the parametric equations of two surfaces in $\mathbb{R}^3(h, k, g)$

$$\Pi_1 : \begin{cases} k = p^2 + h^2 - 4hs + 3s^2 - \dfrac{p^4 - r^4}{4s^2}\\ g = (h - s)s^2 + \dfrac{p^4 - r^4}{4s}, \end{cases} \qquad (s, h) \in \mathbb{R}^2; \tag{16}$$

$$\Pi_2 : \begin{cases} k = -2\lambda^2(h - 2s) - \lambda^4 + \dfrac{r^4}{4s^2}\\ g = \dfrac{1}{2} p^2 (h + \lambda^2) - \lambda^2 s^2 - \dfrac{r^4}{4s}, \end{cases} \qquad (s, h) \in \mathbb{R}^2. \tag{17}$$

Note that the curve (14) is reducible exactly at the values of h, k, g satisfying (7). Therefore it is natural to expect that the bifurcation diagram of the map J belongs to the set defined by equations (7), (16) and (17). The strict proof of this statement is based on the appropriate description of the set of all critical points of (5); this set is organized into four-dimensional manifolds (with singularities) bearing the induced Hamiltonian flows with two degrees of freedom [17]. It is an interesting fact that due to the energy shift in (4) without any mechanical interpretation, equations (16) of the surface Π_1 and equations (7) of the straight lines do not contain the value of the gyrostatic momentum. In particular, equations (16) coincide with the equations of the generalized 4th Appelrot class for the top [5, 18].

Restrict the map J to an arbitrary iso-energetic surface

$$J_h = J|_{E_h}, \qquad E_h = \{\zeta \in P^6 : H(\zeta) = h\}.$$

Then the bifurcation diagram Σ_h of J_h satisfies the above equations of $\Sigma(J)$ with the fixed chosen value h. Obviously, (7) defines two points in the (g, k)-plane. Equations (16) and (17) define some plane curves. As a whole, these curves form the unbounded plane set. In contrast, Σ_h is bounded since any E_h is a compact set. To find the actual boundaries of the curves segments included in Σ_h we need to investigate the intersections $\Pi_1 \cap \Pi_2$ corresponding to the real solutions of (1), (2). The equations of transversal intersections considered on either surface are of degree 5 in h and 12 in s. Even though they can be written down and numerically solved, there is no criterion for the corresponding points to belong to the image of J_h. Finding all special periodic motions gives an appropriate parameterization to the boundaries of the bifurcation diagrams.

3. Analytical solutions

Let f be a smooth function of the complex variables (8). To find its critical points on the submanifold given by (10), (11), it is convenient to use the following equations [5]

$$\begin{aligned}
&\partial_{w_1} f = 0, \qquad \partial_{w_2} f = 0, \qquad \partial_{w_3} f = 0,\\
&\left(2z_2\partial_{x_2} + 2z_1\partial_{y_2} - x_1\partial_{z_1} - y_1\partial_{z_2}\right) f = 0,\\
&\left(2z_1\partial_{x_1} + 2z_2\partial_{y_1} - x_2\partial_{z_2} - y_2\partial_{z_1}\right) f = 0,\\
&\left(x_1\partial_{x_1} - x_2\partial_{x_2} + y_1\partial_{y_1} - y_2\partial_{y_2}\right) f = 0.
\end{aligned} \tag{18}$$

Indeed, six differential operators generating these equations are linearly independent and eliminate the left parts of (10), (11).

It is known that in the case (3) all equilibria of (1) are non-degenerate critical points of the Hamilton function H on P^6 [10]. This means that at these points rank $J = 0$. Hence, rank $J = 1$ yields $\mathrm{d}H \neq 0$ and on periodic solutions both differentials $\mathrm{d}K$ and $\mathrm{d}G$ are proportional to the differential $\mathrm{d}H$; all differentials are calculated on P^6. To formalize this fact, introduce the functions with undefined Lagrange's multipliers σ, τ

$$L_K = K - 2\sigma H, \qquad L_G = 2G - (p^2 - \tau)H$$

and write equations (18) for $f = L_K$ and $f = L_G$. For the function L_K we obtain the equations

$$\begin{aligned}
&\left(w_1^2 + x_1\right)w_2 + \lambda[z_1 + w_1(w_3 - \lambda)] - \sigma w_1 = 0,\\
&(w_2^2 + x_2)w_1 + \lambda[z_2 + w_2(w_3 - \lambda)] - \sigma w_2 = 0,
\end{aligned} \tag{19}$$

$$\lambda w_1 w_2 - \sigma w_3 = 0, \tag{20}$$

$$\begin{aligned}&\big(w_1^2+x_1\big)z_2-\lambda(w_2x_1+w_1y_1)+\sigma z_1=0,\\&\big(w_2^2+x_2\big)z_1-\lambda(w_1x_2+w_2y_2)+\sigma z_2=0,\end{aligned}\tag{21}$$

$$x_1w_2^2-x_2w_1^2+\sigma(y_1-y_2)=0.\tag{22}$$

For L_G the similar system is

$$\begin{aligned}&(\tau-z_1z_2)w_1+x_1y_1w_2+x_1z_2w_3+y_1z_1\lambda=0,\\&x_2y_2w_1+(\tau-z_1z_2)w_2+x_2z_1w_3+y_2z_2\lambda=0,\\&x_2z_1w_1+x_1z_2w_2+(\tau-x_1x_2)w_3+z_1z_2\lambda=0,\\&(x_2z_1+y_2z_2)w_1^2+(x_1z_2+y_1z_1)w_1w_2+(2z_1z_2-x_1x_2)w_1w_3\\&\qquad-x_1y_1w_2w_3-x_1z_2w_3^2+(z_1z_2-\tau)z_1+x_1y_2z_2-\lambda x_1y_1w_2\\&\qquad+\lambda(2z_1z_2-y_1y_2)w_1-\lambda(x_1z_2+y_1z_1)w_3-\lambda^2y_1z_1=0,\\&(x_1z_2+y_1z_1)w_2^2+(x_2z_1+y_2z_2)w_1w_2+(2z_1z_2-x_1x_2)w_2w_3\\&\qquad-x_2y_2w_1w_3-x_2z_1w_3^2+(z_1z_2-\tau)z_2+x_2y_1z_1-\lambda x_2y_2w_1\\&\qquad+\lambda(2z_1z_2-y_1y_2)w_2-\lambda(x_2z_1+y_2z_2)w_3-\lambda^2y_2z_2=0,\\&(\tau-x_1x_2)(y_1-y_2)+x_2z_1^2-x_1z_2^2+2\big(x_2y_2w_1^2-x_1y_1w_2^2\big)\\&\qquad+2(x_2z_1w_1-x_1z_2w_2)w_3+2\lambda(y_2z_2w_1-y_1z_1w_2)=0.\end{aligned}\tag{23}$$

It is shown below that at any critical point of L_K the latter system is consistent automatically for an appropriate value of τ.

First consider the case $\sigma=0$. Then (19)–(22) describe the critical points of K. Equations (19), (20) immediately yield

$$w_1=w_2=0,\qquad z_1=z_2=0.$$

These values satisfy (23) with $\tau=x_1x_2$, meanwhile from (10), (11), $x_1x_2=(a\pm b)^2$. In particular, all critical points of K are also critical points of the function L_G with $\tau=(a\pm b)^2$. Therefore the conditions $\mathrm{d}K=0$, $\mathrm{d}H\neq 0$ yield rank $J=1$. In the initial variables, on the corresponding trajectories we have $\omega_1=\omega_2\equiv 0$, $\alpha_3=\beta_3\equiv 0$. Then the only solutions of (1) are those defined by (6) with the integral constants given by (7). Recall that for $\lambda=0$ the set $\{\mathrm{d}K=0\}\subset P^6$ contains the four-dimensional invariant manifold of the 1st Appelrot class of critical motions in the gravity field (class $K=0$) or, in the case of two fields, its generalization found in the work [13] and studied in [14]. If $\lambda\neq 0$ these solutions have no analogue because the zero value of the integral K is no longer critical and the manifold defined by $\mathrm{d}K=0$ in P^6 is only two-dimensional.

Now suppose that $\sigma\neq 0$. Since the equilibria are already excluded, equation (20) yields $w_1w_2\neq 0$. Denote then

$$w=w_1w_2,\qquad q=w_1/\sqrt{w},\qquad x=x_1x_2.\tag{24}$$

Here $w>0$, $q\in\mathbb{C}$, $|q|=1$, $\overline{q}=1/q$.

Equations (19), (21) provide the linear system in y_1,y_2,z_1,z_2, from which these variables are found as the functions of x_1,x_2,w_1,w_2. Then (22) identically holds. Find w_3 in terms of w_1,w_2 from (20). After that four variables x_1,x_2,w_1,w_2 are subject to three equations following from (10) and (11). As a result in generic case for each arbitrary value of σ we obtain a one-dimensional manifold invariant for equations (9), i.e., a periodic trajectory. Fulfilling this procedure we obtain from (19)–(21)

$$\begin{aligned}
w_3 &= \frac{\lambda}{\sigma} w, \\
z_1 &= -\frac{\sqrt{w}}{\lambda\sigma q}[\sigma x_1 + (\lambda^2+\sigma)q^2(w-\sigma)], \\
z_2 &= -\frac{\sqrt{w}}{\lambda\sigma q}[\sigma q^2 x_2 + (\lambda^2+\sigma)(w-\sigma)], \\
y_1 &= -\frac{(x_1+q^2w)(q^2x_2+w)}{\lambda^2q^2} - \frac{w^2}{\sigma} + \frac{\sigma(\lambda^2+\sigma)}{\lambda^2} - \frac{x_1 w}{\sigma q^2}, \\
y_2 &= -\frac{(x_1+q^2w)(q^2x_2+w)}{\lambda^2q^2} - \frac{w^2}{\sigma} + \frac{\sigma(\lambda^2+\sigma)}{\lambda^2} - \frac{q^2 x_2 w}{\sigma}.
\end{aligned} \tag{25}$$

Use these values to find x_1, x_2 from (10),

$$\begin{aligned}
x_1 &= \frac{r^2\lambda^2\sigma}{(w-\sigma)^2(\lambda^2+\sigma)-\sigma x} - \frac{\lambda^2+\sigma}{\sigma} q^2 w, \\
x_2 &= \frac{r^2\lambda^2\sigma}{(w-\sigma)^2(\lambda^2+\sigma)-\sigma x} - \frac{\lambda^2+\sigma}{\sigma}\frac{w}{q^2}.
\end{aligned} \tag{26}$$

It is convenient to introduce the following notation:

$$u = (w-\sigma)^2(\lambda^2+\sigma) - \sigma x. \tag{27}$$

Substitute (26) and (27) into the last relation (24),

$$2r^2\lambda^2\sigma^2(\lambda^2+\sigma)uwQ = \sigma u^3 + (\lambda^2+\sigma)[\lambda^2w^2+\sigma^2(2w-\sigma)]u^2 + r^4\lambda^4\sigma^4. \tag{28}$$

Here

$$Q = \frac{1}{2}\left(q^2 + \frac{1}{q^2}\right) = \frac{w_1^2+w_2^2}{2w_1w_2} = \cos(2\arg w_1). \tag{29}$$

In addition to (28), the relation between w, Q and u is defined by (11). This relation written at the points (25), (26) takes the form

$$\begin{aligned}
2r^2\lambda^4\sigma^2[(\lambda^2+\sigma)^2u^2 + r^4\lambda^4\sigma^2]uwQ &= [(\lambda^2+\sigma)^2(\lambda^2w+\sigma^2)^2 - 2p^2\lambda^4\sigma^2]u^4 \\
&\quad + r^4\lambda^4\sigma^2[(\lambda^2w+\sigma^2)^2 + (\lambda^2-\sigma)^2\sigma^2 - 4\sigma^4]u^2 + r^8\lambda^8\sigma^6.
\end{aligned}$$

Eliminating Q by (28) we come to the following equation:

$$\begin{aligned}
&\lambda^2(\lambda^2+\sigma)^2u^5 + (\lambda^2+\sigma)[2p^2\lambda^4 - (\lambda^2+\sigma)^3\sigma]\sigma u^4 \\
&\quad + r^4\lambda^6\sigma^2u^3 + 2r^4\lambda^4\sigma^4(\lambda^2+\sigma)^2u^2 - r^8\lambda^8\sigma^6 = 0.
\end{aligned} \tag{30}$$

Thus, for given σ the corresponding value of u is one of the real roots of equation (30), which have constant coefficients as a polynomial in u. Hence u is a constant. After it is determined, we obtain from (28)

$$Q(w) = \frac{\sigma u^3 + (\lambda^2+\sigma)[\lambda^2w^2+\sigma^2(2w-\sigma)]u^2 + r^4\lambda^4\sigma^4}{2r^2\lambda^2\sigma^2(\lambda^2+\sigma)uw} \tag{31}$$

with obvious constraint following from (29)

$$Q^2(w) \leqslant 1. \tag{32}$$

Using algebraic radicals write

$$q(w) = \sqrt{Q + \mathrm{i}\sqrt{1 - Q^2}}, \tag{33}$$

then all phase variables are expressed in terms of one real positive variable w:

$$\begin{aligned}
&w_1 = q(w)\sqrt{w}, \qquad w_2 = \frac{1}{q(w)}\sqrt{w}, \qquad w_3 = \frac{\lambda}{\sigma}w,\\
&x_1 = \frac{r^2\lambda^2\sigma}{u} - \frac{\lambda^2+\sigma}{\sigma}q^2(w)w,\\
&x_2 = \frac{r^2\lambda^2\sigma}{u} - \frac{\lambda^2+\sigma}{\sigma}\frac{1}{q^2(w)}w,\\
&y_1 = \sigma\left(1 + \frac{\sigma}{\lambda^2} - \frac{r^4\lambda^2\sigma}{u^2}\right) + \frac{r^2\lambda^2}{u}q^2(w)w,\\
&y_2 = \sigma\left(1 + \frac{\sigma}{\lambda^2} - \frac{r^4\lambda^2\sigma}{u^2}\right) + \frac{r^2\lambda^2}{u}\frac{1}{q^2(w)}w,\\
&z_1 = \left[\frac{\lambda^2+\sigma}{\lambda}q(w) - \frac{r^2\lambda\sigma}{u}\frac{1}{q(w)}\right]\sqrt{w},\\
&z_2 = \left[\frac{\lambda^2+\sigma}{\lambda}\frac{1}{q(w)} - \frac{r^2\lambda\sigma}{u}q(w)\right]\sqrt{w}.
\end{aligned} \tag{34}$$

The whole system (23) at the values found reduces to one equation

$$\lambda^2\sigma\tau + (\lambda^2+\sigma)u = 0.$$

Given that $\sigma \neq 0$, it is always solvable with respect to τ. Therefore all critical points of the function L_K with $\sigma \neq 0$ are also critical points of the function L_G with an appropriate value of τ. Thus, the points defined by (31), (33), (34) together with the points belonging to the pendulum trajectories (6) completely constitute the set $\{\mathrm{rank}\, J = 1\} \subset P^6$.

To accomplish the description of special periodic motions let us derive the differential equation for $w(t)$. For this purpose, it is enough to consider the last equation in (9). Substituting y_1, y_2 from (34), $q(w)$ from (33) we obtain

$$\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 = \frac{r^4\lambda^2\sigma^2w^2}{u^2}[1 - Q^2(w)]. \tag{35}$$

Using (31) after some obvious transformations we come to the following equation,

$$\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 = \frac{\lambda^2}{4\sigma^2}P_+(w)P_-(w), \tag{36}$$

where

$$\begin{aligned}
&P_\pm(w) = \frac{2\sigma^2r^2}{u}w \pm P(w),\\
&P(w) = w^2 + \frac{2\sigma^2}{\lambda^2}w + \frac{\sigma[u^3 - (\lambda^2+\sigma)\sigma^2u^2 + r^4\lambda^4\sigma^3]}{(\lambda^2+\sigma)\lambda^2u^2}.
\end{aligned} \tag{37}$$

Since $\deg P_+(w)P_-(w) = 4$, equation (36) is integrated in elliptic functions of time. The roots of (37) are explicitly calculated. It makes equation (36) convenient to investigate the existence of solutions. Solve (8) and (34) with respect to the initial real phase variables,

$$
\begin{aligned}
\alpha_1 &= \frac{r^2\lambda^2\sigma - (\lambda^2+\sigma)u}{4r^2\lambda^2\sigma^3(\lambda^2+\sigma)u^2}\left[P(w) - \frac{2r^2\sigma^4}{\lambda^2u^2}(r^2\lambda^2+u)\right],\\
\alpha_2 &= -\frac{r^2\lambda^2\sigma - (\lambda^2+\sigma)u}{4r^2\sigma^3}\sqrt{P_+(w)P_-(w)},\\
\alpha_3 &= \frac{r^2\lambda^2\sigma - (\lambda^2+\sigma)u}{2r\lambda\sigma u}\sqrt{uP_+(w)},\\
\beta_1 &= \frac{r^2\lambda^2\sigma + (\lambda^2+\sigma)u}{4r^2\sigma^3}\sqrt{P_+(w)P_-(w)},\\
\beta_2 &= \frac{r^2\lambda^2\sigma + (\lambda^2+\sigma)u}{4r^2\lambda^2\sigma^3(\lambda^2+\sigma)u^2}\left[P(w) - \frac{2r^2\sigma^4}{\lambda^2u^2}(r^2\lambda^2-u)\right],\\
\beta_3 &= -\frac{r^2\lambda^2\sigma + (\lambda^2+\sigma)u}{2r\lambda\sigma u}\sqrt{uP_-(w)},\\
\omega_1 &= \frac{1}{2r\sigma}\sqrt{uP_+(w)},\\
\omega_2 &= -\frac{1}{2r\sigma}\sqrt{uP_-(w)},\\
\omega_3 &= \frac{\lambda}{\sigma}w.
\end{aligned}
\tag{38}
$$

Together with (36) the obtained expressions give the complete description of all special periodic motions of the Kowalevski gyrostat in two constant fields.

From (38) we easily find that the variable $w > 0$ in the corresponding solutions of (36) must satisfy both inequalities $uP_+(w) \geqslant 0$ and $uP_-(w) \geqslant 0$. Since according to (37) for positive w the values $uP_+(w), uP_-(w)$ cannot be simultaneously negative, it is enough to satisfy the inequality

$$
P_+(w)P_-(w) \geqslant 0. \tag{39}
$$

Finally, we come to the following statement: for given $\sigma \in \mathbb{R}$ the special periodic motions different from (6) exist if and only if there exist $u \in \mathbb{R}$ and $w \in \mathbb{R}, w > 0$ satisfying (30) and (39).

4. The values of the first integrals

To obtain the equations defining the nodes of the diagrams Σ_h, we need to express the first integrals (4) at the points of the above-found solutions. Substitution of (31), (33), (34) into (12) gives the following values of the integral constants:

$$
\begin{aligned}
h_* &= -\frac{u}{2(\lambda^2+\sigma)\sigma} - \frac{(\lambda^2+\sigma)\lambda^2+2\sigma^2}{2\lambda^2} + \frac{(\lambda^2+2\sigma)r^4\lambda^2\sigma^2}{2(\lambda^2+\sigma)u^2},\\
k_* &= -\frac{(\lambda^2+2\sigma)u}{(\lambda^2+\sigma)\sigma} + (\lambda^2+2\sigma)\sigma - \frac{r^4\lambda^4\sigma^3}{(\lambda^2+\sigma)u^2},\\
g_* &= -\frac{(\lambda^2+\sigma)^2u^2 - r^4\lambda^4\sigma^2}{8(\lambda^2+\sigma)^2\lambda^6\sigma^2u^6}[\lambda^4u^6 + (\lambda^2-\sigma)(\lambda^2+\sigma)^2\lambda^2\sigma u^5 - (\lambda^2-2\sigma)(\lambda^2+\sigma)^4\sigma^3u^4\\
&\quad - (\lambda^2+3\sigma)r^4\lambda^6\sigma^3u^3 - 4(\lambda^2+\sigma)^2r^4\lambda^4\sigma^6u^2 + (\lambda^2+2\sigma)r^8\lambda^8\sigma^7].
\end{aligned}
\tag{40}
$$

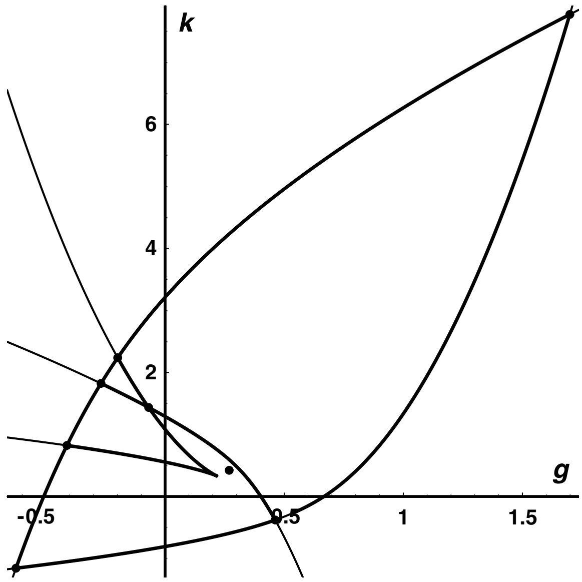

Figure 1. Example of the diagram Σ_h.

Recall that the constant u is connected with σ by equation (30). The expressions provided by (40) can be used to construct computer illustrations of bifurcation diagrams of the map J or its restrictions to iso-energetic levels as the boundary conditions for the two-dimensional sheets or curves respectively. Of course, this process must be preceded by the non-trivial investigation of the existence conditions for the above solutions in terms of the arbitrary constant σ and the physical parameters λ, a, b. Here is the brief scheme. Fix some parameters a, b satisfying (3). Then the pointed out family of periodic solutions depends on two parameters λ and σ. Some obvious degenerations take place if $\lambda = 0$; then we come to the problem of the top motion in two constant fields and the corresponding set splits to the pendulum motions [5] and the special periodic motions of the Bogoyavlensky case [15]. Also the singular cases are $\sigma = 0$ (the case $\mathrm{d}K = 0$ studied above) and $\lambda^2 + \sigma = 0$ (the above formulae have some singularities needing extra investigation). More bifurcations of the family happen when the parameters cross the values producing a multiple root of the polynomial $P_+(w)P_-(w)$. It is easy to see that, if λ and σ differ from the values already discussed, neither P_+ nor P_- have multiple roots. Their common root may be only zero. Eliminating u between the equations $P_\pm(0) = 0$ and (30) we obtain the separating set in the plane (λ, σ); this set corresponds to the existence of the gyrostat equilibria when $w(t) \equiv 0$. Furthermore it is necessary to investigate the cases of existence of a multiple root of equation (30) in u; such a root must provide at least one real solution of (36). Contemporary systems of symbolic computations give the possibility of obtaining some conditions for the parameters and create an effective algorithm calculating the nodes of bifurcation diagrams for any given energy value. The example of the bifurcation diagram Σ_h cut by the points (40) from the curves (16), (17) is shown in figure 1. Here $a = 1, b = 0.35, \lambda = 0.25, h = 0.8$. We see that the values (7) can give an isolated point in Σ_h as in the case shown, or a point of self-intersection of the curves (16), (17) similar to the case $\lambda = 0$ [5]. The values (g_*, k_*) found from (40) and the condition $h_* = h$ correspond to transversal intersections of Π_1 and Π_2.

We now prove directly that the points defined by (40) belong to both surfaces Π_1, Π_2. For the surface Π_1 put

$$u = \frac{r^2\lambda^2\sigma}{R} \qquad \left(R = \sqrt{(\lambda^2+\sigma)^2 + 2\lambda^2 s}\right). \tag{41}$$

Then equations (40) yield

$$\begin{aligned}
h_* &= \frac{(\lambda^2+2\sigma)s}{\lambda^2+\sigma} + \sigma - \frac{r^2\lambda^2}{2(\lambda^2+\sigma)R},\\
k_* &= -\frac{2\lambda^2\sigma s}{\lambda^2+\sigma} + \sigma^2 - \frac{(\lambda^2+2\sigma)r^2\lambda^2}{(\lambda^2+\sigma)R},\\
g_* &= \frac{s}{2(\lambda^2+\sigma)^2}\left\{\frac{r^4\lambda^4}{2[2\lambda^2 s + (\lambda^2+\sigma)^2]} + 2[(\lambda^2+2\sigma)s + (\lambda^2+\sigma)^2]\sigma s\right.\\
&\qquad \left. - \frac{r^2[(\lambda^2+3\sigma)\lambda^2 s + 2(\lambda^2+\sigma)^2\sigma]}{R}\right\}.
\end{aligned} \tag{42}$$

It is necessary to satisfy equation (30). Substitute the value of u into it by (41),

$$r^2[\lambda^2 s + (\lambda^2+\sigma)^2] - [2\sigma s^2 + p^2(\lambda^2+\sigma)]R = 0. \tag{43}$$

Eliminating R in (42) by (43), we obtain the values of h, k, g satisfying (16). Hence, any point (40) belongs to Π_1.

For the surface Π_2 put

$$u = -2\lambda^2\sigma s, \tag{44}$$

then from (40)

$$\begin{aligned}
h_* &= \frac{8\lambda^4 s^3 - 4(\lambda^2+\sigma)[(\lambda^2+\sigma)\lambda^2 + 2\sigma^2)]s^2 + (\lambda^2+2\sigma)r^4}{8(\lambda^2+\sigma)\lambda^2 s^2},\\
k_* &= \frac{8(\lambda^2+2\sigma)\lambda^2 s^3 + 4(\lambda^2+\sigma)(\lambda^2+2\sigma)\sigma s^2 - r^4\sigma}{4(\lambda^2+\sigma)s^2},\\
g_* &= -\frac{4(\lambda^2+\sigma)^2 s^2 - r^4}{512(\lambda^2+\sigma)^2\lambda^6 s^6}[64\lambda^8 s^6 - 32(\lambda^2-\sigma)(\lambda^2+\sigma)^2\lambda^4 s^5 - 16(\lambda^2-2\sigma)(\lambda^2+\sigma)^4\sigma s^4\\
&\qquad + 8(\lambda^2+3\sigma)r^4\lambda^4 s^3 - 16(\lambda^2+\sigma)^2 r^4\sigma^2 s^2 + (\lambda^2+2\sigma)r^8\sigma].
\end{aligned} \tag{45}$$

Substitute h, k by h_*, k_* in the first equation (17). It becomes the identity. Equation (30) with (44) yields

$$\begin{aligned}
&32(\lambda^2+\sigma)^2\lambda^4 s^5 - 16(\lambda^2+\sigma)[2p^2\lambda^4 + (\lambda^2+\sigma)^3\sigma]s^4\\
&\qquad + 8r^4\lambda^4 s^3 - 8(\lambda^2+\sigma)^2 r^4\sigma s^2 + r^8\sigma = 0.
\end{aligned} \tag{46}$$

Using the values h_*, g_* from (45) calculate the expression

$$4sg_* + r^4 - 2sp^2(h_* + \lambda^2) + 4\lambda^2 s^3. \tag{47}$$

It turns out to be the left part of equation (46) multiplied by

$$\frac{8\lambda^4 s^3 + 4(\lambda^2-2\sigma)(\lambda^2+\sigma)^2 s^2 + (\lambda^2+2\sigma)r^4}{128(\lambda^2+\sigma)^2\lambda^6 s^5}.$$

Therefore, (47) is equal to zero and the second equation (17) holds.

Thus, the values of the first integrals at the pointed out family of periodic motions provide the parametric equations of the transversal intersections of the bifurcation sheets corresponding to real solutions in the problem of motion of the Kowalevski gyrostat in two constant fields.

5. Conclusion

In this paper we have considered the case of motion about a fixed point of a gyrostat with the inertia tensor of the Kowalevski type under the action of two independent constant force fields described by the completely integrable irreducible Hamiltonian system with three degrees of freedom [1]. No explicit integration of this system or its subsystems with non-zero gyrostatic momentum and asymmetric combination of the forces is known yet. The main results can be summarized as follows.

The equations of the surfaces bearing the bifurcation diagram of the integral map are derived from the known Lax representation. To cut out the bifurcation diagram itself, one needs to find all the cases when the rank of the integral map is less than 2. These cases correspond either to the points of equilibria, which are easily calculated, or to the so-called special periodic motions. Such motions are of great interest in the problem of topological classification because they define the bifurcations inside the existing critical subsystems with two degrees of freedom and generate the nodes of the bifurcation diagrams induced on iso-energetic surfaces. The knowledge of the nodes is necessary to apply various methods of finding topological invariants of the system. This problem is solved here by obtaining the explicit formulae (6) and (38) for all special periodic motions of the Kowalevski gyrostat in two constant fields. It follows from (6) and (36) that the phase variables are elliptic functions of time. Equations (40) represent the values of the first integrals in terms of the physical parameters of the gyrostat and the force fields and two additional parameters satisfying one algebraic restriction. These values are also expressed in terms of the parameters on the smooth sheets of the bifurcation diagram.

The obtained results allow us to complete in the nearest future the classification of the bifurcation diagrams with respect to the physical parameters started in [17, 19] and provide the necessary basis for the topological analysis of this highly complicated integrable system.

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